

Time-optimal bath-induced unitaries by Zermelo navigation: speed limit and non-Markovian quantum computation

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The solution of the quantum Zermelo navigation problem is applied to the non-Markovian open system dynamics of a set of quantum systems interacting with a common environment. We consider a case allowing an exact time-optimal realization of environment-mediated non-local system unitaries. For a linear coupling to a harmonic bosonic bath, we derive a speed limit for the implementation time in terms of the fundamental frequency of the bath modes. As a product of two exponentials of the local free wind and the pairwise system-coupling, the Zermelo unitary forms a natural building block for reaching a general unitary by concatenation.

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I. INTRODUCTION

The discussion of optimal control protocols for evolving quantum systems and the relevance of transition speeds and their limits can be traced back to the dawn of quantum theory as can be seen in the example of Fermi's golden rule for the mean transition rate into an orthogonal energy eigenstate. While touching fundamental problems about computation *per se* [1–3], it is also of technological interest, since minimization of computing time amounts to minimizing the interaction time with detrimental environmental noise sources. The speed optimization can refer to the transformation of a given initial to a target state, in particular the evolution into an (arbitrary) orthogonal state, or it can refer to the implementation of a desired target unitary, which is supposed to act on an arbitrary (unknown) quantum state. The existence of a physical solution to the optimization problem and the formulation of finite speed limits generally require a set of preset constraints which represent practical limitations encountered. They can be of energetic nature restricting the size of the Hamiltonians in some way or refer to the type (subspace) of Hamiltonians that one can implement. The latter kind of constraints can be added by Lagrange multipliers [4–6] giving rise to quantum brachistochrone curves as optimal trajectories of states or unitaries.

Besides of this, a fundamental constraint addresses external background fields that cannot be manipulated and give rise to a free (natural) Hamiltonian \hat{H}_0 , which must be taken as given. Even in the absence of restrictions on the implementable control Hamiltonians $\hat{H}_C(t)$ (which would suggest to simply substitute $\hat{H}'_C = \hat{H}_C - \hat{H}_0$), the mentioned energetic constraints limit the size of implementable control fields, hence the generic \hat{H}_0 cannot be ignored. This is the motivation for the quantum Zermelo navigation problem which, inspired by a result in [7], was formulated in [8] and recently solved in [5, 6].

Roughly speaking, external forces as caused by a background field are incorporated into the problem description as a geometric property by defining an appropriate

measure of distance with respect to which one then freely takes a shortest path. In the example of the Zermelo navigation problem, the solution consists in following the geodesics of a Randers metric, which is a special type of Finsler metric and can be derived from Riemann metrics by adding a linear term. We refer to [5, 6] and references therein for a description of the mathematical background.

We here consider the navigation of unitary operators. Starting from a given initial unitary \hat{U}_I , the task is to find a time-dependent Hamiltonian $\hat{H}(t)$ that implements a desired final unitary \hat{U}_F in shortest possible time. The total Hamiltonian

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_C(t) \quad (1)$$

is assumed to consist of a constant part \hat{H}_0 that cannot be altered and a time-dependent part $\hat{H}_C(t)$ which can be controlled by the experimenter without limits except that the size of the control is bounded, so that we set a fixed $\text{Tr}[\hat{H}_C^2(t)]$.

The reference to Zermelo originates in the classical problem of finding a time-dependent heading of a ship or aircraft that, starting from some initial location, navigates it to a destination in shortest possible time [9]. \hat{H}_0 here corresponds to a wind or current, while a constant $\text{Tr}[\hat{H}_C^2(t)]$ corresponds to the assumption of “full speed ahead” at any time. In the classical problem, the optimal route under constant wind or current is a straight line from start to finish, which is achieved by keeping a constant correction angle of the vessel's heading with respect to its destination to compensate the drift off. More general versions of the classical Zermelo navigation problem that include obstacles and a wind depending on location and time have also been of recent interest in animal behavior research [10] and computer science [11].

Contrary to intuition, the solution of the corresponding quantum problem is *not* a constant $\hat{H}_C(t) \equiv \hat{H}_C$ which would give rise to a (single-exponential) one-parameter subgroup of unitaries $\hat{U}(t) = \text{Te}^{-i \int_0^t d\tau \hat{H}(\tau)} = e^{-i(\hat{H}_0 + \hat{H}_C)t}$, but instead an explicit time-dependence such that \hat{H} be-

comes time-independent in the interaction picture [5, 6],

$$\hat{H}_C(t) = e^{-i\hat{H}_0 t} \hat{H}_C(0) e^{i\hat{H}_0 t}, \quad (2)$$

and for which the time-optimal curve of unitaries disentangles according to $\hat{U}(t) = \text{Te}^{-i\int_0^t d\tau \hat{H}(\tau)} = e^{-i\hat{H}_0 t} e^{-i\hat{H}_C(0)t} = e^{-i\hat{H}_C(t)t} e^{-i\hat{H}_0 t}$ into two exponentials. (Here, T denotes positive time ordering, and we have set $\hbar=1$.) Time-independence as in the classical case is thus only recovered for $[\hat{H}_0, \hat{H}_C(0)] = 0$.

Zermelo navigation of quantum *states* has been solved in [12] also deriving a limit on the passage time, by tracing it back to the navigation of unitaries. The energetic constraint here takes the form of a fixed variance $\langle \Delta \hat{H}_C^2 \rangle$. The special case of time-independent \hat{H}_C has been discussed in [13], placing focus on two-level systems and demonstrating that the treatment is astonishingly non-trivial even with these restrictions. Interestingly, Eq.(5) in [12] transforms the state navigation problem $|\Psi_F\rangle = \hat{U}(t)|\Psi_I\rangle$ (in its general form) into that of hitting a (counter)moving target, $e^{-i\hat{H}_C(0)t}|\Psi_I\rangle = e^{i\hat{H}_0 t}|\Psi_F\rangle$, which suggests a close connection to pursuit problems: it advises to move “straightly” to where the target will be at the time of arrival. This is analogous to the classical case of a dog who should straightly move towards the extrapolated target (her walking master’s) location, rather than intuitively following her snout’s direction heading towards the instantaneous location of the target resulting in a longer so-called “Hundekurve”.

Referring to time-independent systems with Hamiltonian \hat{H} , [14] relates speed limits for the implementation of unitaries to a family of bounds on orthogonality times, which are all of the canonical form $t_\perp \geq \frac{\pi\hbar}{2E}$, including the Margolus-Levitin theorem [15] ($E = \langle \psi | \hat{H} | \psi \rangle - E_0$, E_0 being the lowest eigenvalue of \hat{H}), while relation to the time-energy uncertainty relation (Mandelstam-Tamm bound [16], $E = \sqrt{\langle \psi | \hat{H}^2 | \psi \rangle - \langle \psi | \hat{H} | \psi \rangle^2}$) is also discussed.

Here, we show how the Zermelo navigation (2) can be applied to a case of non-Markovian evolution of open quantum systems [17] such that it allows an exact treatment of a set of quantum systems coupled to a common environment. In contrast to the original motivation for realizing a time-optimal evolution, namely to minimize the temporal accumulation of environment-induced decoherence effects, we demonstrate that Zermelo time-optimality can enable a *deliberate* environmental coupling, which here serves to mediate a desired system coupling. This intended utilization of non-Markovian (memory) properties of the environment thus contrasts the case, where environmental interactions are accompanied with irreversible decoherence. Bath-induced system interactions have been discussed for several years, cf. e.g. [18, 19]. The main focus of these Markovian approaches has been on the preparation of entangled many-body steady states [20–23] for dissipative quantum computation [24] and continuous quantum repeaters [25].

In contrast, here we consider a navigation of unitaries acting on a set of systems and a shared bath. Starting from the identity, $\hat{U}_I = \hat{I}$, the navigation periodically passes unitaries $\hat{U}(t_m)$ in which system and bath parts are refactorized. Although the systems never interact directly, the system part of the target unitary contains an environment-induced pairwise system-coupling which is decoherence-free. All results are independent of the system and bath states and apply exact analytical expressions which do not rely on assumptions such as the weak coupling perturbative approach, rotating wave or Markovian approximations.

The present work builds on [26–28], who demonstrate the preparation of nonclassical states such as the GHZ-state, cf. also [29]. However, the scheme presented there requires setting $\hat{H}_0 = 0$ to allow for an analytic solution. Below we will show that (a) it is remarkably the Zermelo navigation (2) that enables us to overcome this limitation, allowing for arbitrary \hat{H}_0 . As a consequence, the described protocol of generating bath-mediated system-coupling is time-optimal, a result thus also holding for the schemes in [26–28] as a special case of vanishing \hat{H}_0 . Based on this, we construct (b) a *state-independent* speed limit for the refactorization time. Finally, we address the question of reaching an arbitrarily given target \hat{U}_F . While a single Zermelo navigation $\hat{U}(t)$ allows in our case merely the creation of bath-induced pairwise unitary system interactions, the fact that $\hat{U}(t)$ is a product of two non-commuting exponentials suggests the reachability of a given \hat{U}_F by concatenation as realized by Hamiltonian resets. The two-exponential form of $\hat{U}(t)$ thus halves the number of required resets. We will (c) provide a numerical proof of principle demonstration, where the systems are three two-level systems (“qubits”), and \hat{U}_F is a Toffoli gate. Apart from being optimal for quantum error correction, the Toffoli gate alone is universal for reversible computing and, together with the Hadamard gate, forms a universal set of gates for quantum computing [30]. While the three qubit-setup here serves as simplest case going beyond pairwise system interactions, it is hence already sufficient as building block for universal quantum computation.

This work is organized as follows. After this introduction, the realization of an exact decoherence-free bath-mediated unitary system coupling by means of Zermelo navigation is presented in Sec. II. Sec. IIA defines the general setting and conditions, Sec. IIB introduces the example of a linear coupling to a bosonic bath, for which the speed limit is derived in Sec. IIC. A numerical approach to the problem of navigating to a general target is given in Sec. III. As a setup relevant for quantum computing, the special case of three qubits is considered for a Toffoli gate in Sec. IIIA and for a repeater relay station in Sec. IIIB. An approach to the navigation to a general target that does not rely on the requirements of the previous sections is outlined in Sec. IV. Finally, a summary and outlook on future work is provided in Sec. V.

II. BATH-INDUCED UNITARIES

A. Zermelo navigation for bath-coupled systems

Consider N systems $j = 1, \dots, N$ and one bath B as depicted in Fig. 1 with factorized individual system-bath

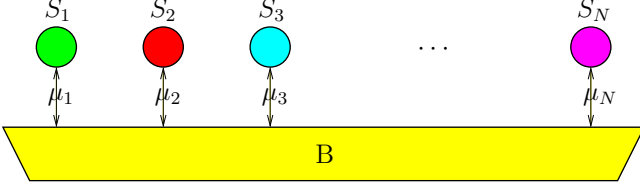


FIG. 1: A set of systems S_j interacting with a shared bath (environment) B. An extension with modulated system-bath interaction strengths, where each term in (4) is multiplied with a $\mu_j(t)$ will be discussed in Sec. III below.

interactions,

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{SB}, \quad (3)$$

$$\hat{H}_S = \sum_{j=1}^N \hat{H}_j, \quad \hat{H}_{SB} = \sum_{j=1}^N \hat{S}_j \otimes \hat{B}_j, \quad (4)$$

where \hat{S}_j are Hermitian operators of system j (hence $[\hat{S}_j, \hat{S}_k] = 0$) and \hat{B}_j are Hermitian operators of the bath (hence $[\hat{B}_j, \hat{B}_k] \neq 0$).

The division in the quantum Zermelo navigation problem of the Hamiltonian (1) into an uncontrollable “wind” part \hat{H}_0 and a controllable part \hat{H}_C is not fixed and refers to the operators one can and wants to alter. In our case (3) of a composite quantum system, a natural assumption is that one can modify operators of the system but not those of the bath. We may thus identify in (3)

$$\hat{H}_0 = \hat{H}_S, \quad \hat{H}_C = \hat{H}_B + \hat{H}_{SB}, \quad (5)$$

for which (2) with $\hat{H}_C(0) \equiv \hat{H}_C$ becomes

$$\hat{H}_C(t) = \hat{H}_B + \sum_{j=1}^N \hat{S}_j(t) \otimes \hat{B}_j, \quad (6)$$

where $\hat{S}_j(t) \equiv \hat{U}_j(t) \hat{S}_j \hat{U}_j^\dagger(t)$ with $\hat{U}_j(t) \equiv e^{-i\hat{H}_j t}$. We see that only the individual system operators \hat{S}_j have to be navigated in their respective local winds \hat{H}_j , whereas all bath operators \hat{H}_B and \hat{B}_j remain unaltered.

Given two general non-commuting and explicitly time-dependent Hamiltonians $\hat{H}_1(t)$ and $\hat{H}_2(t)$, defining $\hat{U}_1(t_2, t_1) \equiv \text{Te}^{-i \int_{t_1}^{t_2} dt \hat{H}_1(t)}$, and using the convention $\hat{U}_1(t_1, t_2) = \hat{U}_1^\dagger(t_2, t_1)$, we can disentangle the time-

ordered product according to [31]

$$\hat{U}(t_2, t_1) \equiv \text{Te}^{-i \int_{t_1}^{t_2} dt [\hat{H}_1(t) + \hat{H}_2(t)]} \quad (7)$$

$$= \hat{U}_1(t_2, t_1) \text{Te}^{-i \int_{t_1}^{t_2} dt \hat{U}_1(t_1, t) \hat{H}_2(t) \hat{U}_1(t, t_1)} \quad (8)$$

$$= \hat{U}_1(t_2, t_3) \text{Te}^{-i \int_{t_1}^{t_2} dt \hat{U}_1(t_3, t) \hat{H}_2(t) \hat{U}_1(t, t_3)} \hat{U}_1(t_3, t_1), \quad (9)$$

where the auxiliary time t_3 can be chosen as desired. Applying (8) with $t_1 = 0$, $t_2 = t$, $\hat{H}_1 = \hat{H}_S + \hat{H}_B$, $\hat{H}_2(\tau) = \sum_{j=1}^N \hat{S}_j(\tau) \otimes \hat{B}_j$ gives for the total evolution operator $\hat{U}(t) \equiv \hat{U}(t, 0)$

$$\begin{aligned} \hat{U}(t) &= \text{Te}^{-i \int_0^t d\tau \hat{H}(\tau)} \\ &= \hat{U}_S(t) \hat{U}_B(t) \hat{U}_{SB}(t), \end{aligned} \quad (10)$$

$$\hat{U}_{SB}(t) = \text{Te}^{-i \int_0^t d\tau \hat{H}_{SB}(\tau)} \equiv e^{-it \hat{H}_{\text{eff}}(t)}, \quad (11)$$

where $\hat{U}_S(t) \equiv e^{-i\hat{H}_S t}$, $\hat{U}_B(t) \equiv e^{-i\hat{H}_B t}$ and $\hat{H}_{SB}(\tau) \equiv \sum_{j=1}^N \hat{S}_j \otimes \hat{B}_j(\tau)$ with $\hat{B}_j(\tau) \equiv \hat{U}_B^\dagger(\tau) \hat{B}_j \hat{U}_B(\tau)$. [In physical terms, this dynamic time dependence of $\hat{H}_{SB}(\tau)$ via $\hat{B}_j(\tau)$ refers to the Dirac interaction picture and must be distinguished from the explicit time dependence of $\hat{H}_C(t)$ via $\hat{S}_j(t)$ introduced in (6). The latter refers to the Schrödinger picture and describes the actual navigation carried out by the experimenter.]

Eq. (11) defines some Hermitian operator \hat{H}_{eff} , which can be expressed by means of the Magnus series expansion [32]. \hat{H}_{eff} itself is time-dependent, and we have left a factor t for dimensional reasons. Since

$$[\hat{H}_{SB}(t_1), \hat{H}_{SB}(t_2)] = \sum_{j,k=1}^N \hat{S}_j [\hat{B}_j(t_1), \hat{B}_k(t_2)] \hat{S}_k, \quad (12)$$

in the special case when $[\hat{B}_j(t_1), \hat{B}_k(t_2)]$ are c -number functions, only the first two terms of the Magnus series are nonzero, and we obtain a closed expression

$$\begin{aligned} \hat{H}_{\text{eff}}(t) &= \frac{1}{t} \int_0^t d\tau \hat{H}_{SB}(\tau) \\ &\quad - \frac{i}{2t} \int_0^t dt_1 \int_0^{t_1} dt_2 [\hat{H}_{SB}(t_1), \hat{H}_{SB}(t_2)] \end{aligned} \quad (13)$$

$$= \sum_{j=1}^N \hat{S}_j \otimes \hat{F}_j(t) + \sum_{j,k=1}^N \hat{S}_j \kappa_{jk}(t) \hat{S}_k, \quad (14)$$

where

$$\hat{F}_j(t) = \frac{1}{t} \int_0^t d\tau \hat{B}_j(\tau), \quad (15)$$

$$\kappa_{jk}(t) = -\frac{i}{2t} \int_0^t dt_1 \int_0^{t_1} dt_2 [\hat{B}_j(t_1), \hat{B}_k(t_2)]. \quad (16)$$

B. Linear coupling to a bosonic bath

A simple example in which the Magnus series breaks off at second order is a bath of bosonic modes [26–28].

The bath parts of \hat{H}_{SB} in (4)

$$\hat{B}_j = \sum_l A_{jl} \hat{b}_l + h.a. \quad (17)$$

are linear combinations of the bosonic mode operators \hat{b}_l and $h.a.$ denotes the Hermitian conjugate. A bath Hamiltonian

$$\hat{H}_{\text{B}} = \sum_l \omega_l \hat{b}_l^\dagger \hat{b}_l \quad (18)$$

then gives the time evolution $\hat{B}_j(t) = \sum_l A_{jl} e^{-i\omega_l t} \hat{b}_l + h.a.$ Using $[\hat{b}_l, \hat{b}_{l'}^\dagger] = \delta_{ll'}$ we can verify that the above-mentioned c -number condition is fulfilled, $[\hat{B}_j(t_1), \hat{B}_k(t_2)] = 2i \text{Im} \sum_l A_{jl} A_{kl}^* e^{-i\omega_l(t_1-t_2)}$. If the A_{jl} couple only to harmonics of some given frequency ω , i.e., they are nonzero only for $\omega_l = l\omega$, $l \in \mathbb{N}$, then for discrete times

$$t_m = m\Delta t, \quad \Delta t = \frac{2\pi}{\omega} \quad (19)$$

($m \in \mathbb{N}$) the integral (15) vanishes, $\hat{F}_j(t_m) = 0$, whereas (16) reduces to $\kappa_{jk} \equiv \kappa_{jk}(t_m) = -\text{Re} \sum_l A_{jl} A_{kl}^* / (l\omega)$. Combining the A_{jl} to a matrix \mathbf{A} and defining a diagonal matrix $\mathbf{\Omega}$ with elements $\Omega_{ll'} = -\delta_{ll'} / (l\omega)$ we can write $\kappa_{jk} = \text{Re}(\mathbf{A}\mathbf{\Omega}\mathbf{A}^\dagger)_{jk}$ and combine the κ_{jk} to a matrix $\mathbf{\kappa}$.

In summary, the total evolution operator (10) factorizes at the discrete times t_m according to

$$\hat{U}(m\Delta t) = e^{-im\Delta t \hat{H}_{\text{S}}} e^{-im\Delta t \hat{\mathbf{S}} \cdot \mathbf{\kappa} \cdot \hat{\mathbf{S}}} \otimes e^{-im\Delta t \hat{H}_{\text{B}}}, \quad (20)$$

where $\hat{\mathbf{S}} \cdot \mathbf{\kappa} \cdot \hat{\mathbf{S}} = \sum_{j,k=1}^N \hat{S}_j \kappa_{jk} \hat{S}_k$. The first two exponentials are a result of the Zermelo navigation, whereas the last one describes the free evolution of the decoupled bath.

C. Speed limit

In our scenario, a set of N systems interact with a common bath such that the overall unitary evolution factorizes for discrete times (19) into system and bath parts. The bath nevertheless affects the system evolution by means of the exponential containing the \hat{S}_j in (20). This is in contrast to Markovian bath effects [17], where information dissipates irreversibly into the environment thus preventing the realization of a unitary transformation. Unlike the system evolution, the evolution of the bath itself in (20) is not affected. The Zermelo navigation serves to remove any time dependence from the control Hamiltonian in the interaction picture which allows a straightforward evaluation of the obtained target unitary by means of the Magnus identity. Due to the need of a well-defined coupling, the bath here acts as an ancillary system such as a field confined by a cavity rather than an uncontrolled environment.

Obviously, $\text{Tr}[\hat{H}_{\text{C}}^2(t)] = \text{Tr}_1 \dots \text{Tr}_N \text{Tr}_B[\hat{H}_{\text{C}}^2(t)]$ is undetermined but fixed because it is time-independent. In

contrast to the finite-dimensional systems for which the quantum Zermelo navigation problem has been discussed so far, the bath considered here is an infinite set of infinite-dimensional systems. The Zermelo solution of the problem does not contain any reference to the system size, however, which suggests that it holds in our more general setting too, cf. also a comment in [6]. (We may truncate the bath with respect to the number of relevant bath modes and at some sufficiently high occupation number for each of these modes without changing any physical properties of the model.) In (20), the infinite bath is decoupled from the unitary evolution of the N systems, where the latter may be finite or infinite dimensional. Restricting the modulation (imposed time-dependency) to the systems renders the method independent on any need to manipulate the bath.

Since we apply Zermelo navigation, the described method of bath-induced realization of the above type of unitaries is time-optimal with respect to the local “winds” \hat{H}_j . Substituting $\omega = \frac{2E_0}{\hbar}$, where E_0 is the ground state energy of a harmonic oscillator [i.e., if we set $\hat{H}_{\text{B}} \equiv \hbar\omega(\hat{n} + \frac{1}{2})$], the minimum time required for refactorization of the unitary into system and bath parts can be written in the canonical form mentioned in the beginning

$$t_{\min} = \frac{\pi\hbar}{E_0}. \quad (21)$$

A single harmonic oscillator with fundamental frequency ω here serves as a minimum system suitable to act as bath in the described way. This minimal implementation time is finite but small for large E_0 . An observer on a large time scale, on which she cannot resolve t_{\min} , may thus be unaware of it and even the existence of a background bath. Such an observer would simply witness a time-continuous evolution of the systems, whose bath-mediated interactions would exhibit as direct pairwise system interactions.

III. CONCATENATIONS

So far we have ignored the problem of reaching a given target unitary. In [5], a method is presented to calculate the initial $\hat{H}_{\text{C}}(0)$ such that $\hat{U}(t)$ after some time t reaches the target \hat{U}_{F} , $\hat{U}(t) = \hat{U}_{\text{F}} \hat{U}_{\text{I}}^\dagger$, so that $\hat{U}(0)\hat{U}_{\text{I}} = \hat{U}_{\text{I}}$ and $\hat{U}(t)\hat{U}_{\text{I}} = \hat{U}_{\text{F}}$. In our context, we face the additional restriction to pairwise couplings $\hat{\mathbf{S}} \cdot \mathbf{\kappa} \cdot \hat{\mathbf{S}}$ in (20). The system part of (20) is itself a product of two factors, $\hat{U}_{\text{system}}(m\tau) = e^{-im\tau \hat{B}} e^{-im\tau \hat{A}}$, given by $\hat{A} = \sum_{j,j'=1}^N \hat{S}_j \kappa_{jj'} \hat{S}_{j'}$ and $\hat{B} = \sum_{j=1}^N \hat{H}_j$, respectively, where τ is fixed by (19). This restricts the set of implementable N -system target unitaries $\hat{U}_{\text{system}}(t_m = m\tau) \stackrel{!}{=} \hat{U}_{\text{F}}$, even if the \hat{S}_j can be chosen at will. To generalize the set of reachable \hat{U}_{F} , we may concatenate individual Zermelo navigations by adjusting at times $t_m = \sum_{k=1}^m \tau_k$ [by means of ω , cf. (19)] τ to τ_{m+1} , $\kappa_{jj'}$ to $\kappa_{jj'}^{(m+1)}$ [by multiplying each term in \hat{H}_{SB}

in (4) with a system-bath interaction strength $\mu_j^{(m+1)} \geq 0$, and reset the $\hat{S}_j(t)$ to \hat{S}_j . This gives $\hat{U}_{\text{system}}(t_m) = \prod_{k=1}^m e^{-i\beta_k \hat{B}} e^{-i\alpha_k \hat{A}_k^{(\mathcal{N})}}$ with $\hat{A}_k^{(\mathcal{N})} = \|\hat{A}_k\|^{-1} \hat{A}_k$, $\alpha_k = \|\hat{A}_k\| \tau_k$, $\beta_k = \tau_k$, and given $\hat{A}_k = \sum_{j,j'=1}^N \hat{S}_j \kappa_{jj'}^{(k)} \hat{S}_{j'}$ with $\kappa_{jj'}^{(k)} = \mu_j^{(k)} \mu_{j'}^{(k)} (\tau_k / \tau) \kappa_{jj'}$. Here, we factored out $\|\hat{A}\|^2 \equiv \text{Tr}(\hat{A}^\dagger \hat{A})$ for convenience, and without loss of generality we assume that $\|\hat{B}\| = 1$. The special case $\mu_j^{(k)} \equiv \mu^{(k)}$ and hence $\kappa_{jj'}^{(k)} \sim \kappa_{jj'}$ generates an alternate product for which $\hat{A}_k^{(\mathcal{N})} \equiv \hat{A}^{(\mathcal{N})}$, which can approximate any $\hat{U}_F = e^{-i\hat{H}}$, with \hat{H} being a member of the algebra spanned by the multi-commutators from \hat{A} and \hat{B} [33], where the α_k and β_k can be determined from \hat{U}_F [34]. The resulting concatenated navigation represents a (piecewise differentiable) evolution from \hat{I} to \hat{U}_F under the constraint of fixed $\hat{A}^{(\mathcal{N})}$ (or $\hat{A}_k^{(\mathcal{N})}$) and \hat{B} . The Zermelo-optimality holds only for the segments $e^{-i\beta_k \hat{B}} e^{-i\alpha_k \hat{A}_k^{(\mathcal{N})}}$, during which the two factors $e^{-i\alpha_k \hat{A}_k^{(\mathcal{N})}}$ and $e^{-i\beta_k \hat{B}}$ are hence implemented at once within τ_k rather than in consecution, sparing the need to switch between the generators \hat{A}_k and \hat{B} .

The number N of systems can be arbitrary including infinite. In the fundamental case $N = 1$, in which no division into subsystems is assumed, the effect of the bath reduces to “dressing” the system evolution, $\hat{A}_k \sim \hat{S}^2$. For $N > 1$, the individual systems j and their bath-coupling as defined by \hat{H}_j and \hat{S}_j can be set arbitrarily and do not need to be the same for each j . For $N > 2$, one may argue that even if we can choose the \hat{S}_j at will, they are local operators as are the \hat{H}_j . However, since \hat{A} (or \hat{A}_k) is 2-local (i.e., it represents a pairwise system-coupling), and each commutator of a k -local operator with \hat{A} generally yields a $(k+1)$ -local operator unless the number of systems has been reached, the concatenation builds up N -system unitaries, analogous to a quantum circuit consisting of one- and two-qubit gates.

A. Quantum circuits with qubits

A “continuous variable” example is a set of harmonic oscillators with parametric-amplifier type couplings, $\hat{S}_j = \hat{n}_j$, for which the bath induces Kerr and cross-Kerr nonlinearities $\hat{S}_j \hat{S}_k = \hat{n}_j \otimes \hat{n}_k$. Of special interest for quantum information processing are two-level systems, however, for which we now provide some explicit examples. If \hat{H}_0 and \hat{H}_C are given appropriately, \hat{U}_F may represent a desired gate without need of concatenations. An example for $N=2$, where \hat{H}_0 and \hat{H}_C commute, is the CNOT-gate [35] $\hat{U}_{\text{CNOT}} = e^{-i\frac{\pi}{4}} e^{i\frac{\pi}{4} \hat{\sigma}_1^{(2)}} e^{i\frac{\pi}{4} \hat{\sigma}_3^{(1)}} e^{-i\frac{\pi}{4} \hat{\sigma}_3^{(1)} \hat{\sigma}_1^{(2)}}$, which can be implemented if $\hat{S}_1 = -\hat{H}_1 = \hat{\sigma}_3^{(1)}$, $\hat{S}_2 = -\hat{H}_2 = \hat{\sigma}_1^{(2)}$, $\tau = \frac{\pi}{4}$, and $\kappa_{12} = \frac{1}{2}$.

If the \hat{H}_j and \hat{S}_j are fixed by the experimental setup, concatenations are required to implement a desired \hat{U}_F . As a concrete example, we consider three qubits with ran-

domly given \hat{H}_j and \hat{S}_j , whose interaction strengths with a shared bosonic bath can be altered [by multiplying each term in \hat{H}_{SB} in (4) with a factor $\mu_j^{(k)} \geq 0$ over time interval τ_k as mentioned above], and where the bosonic bath is eliminated by Zermelo-navigation of the \hat{S}_j leading to products $\hat{U}_{\text{system}}(t_m) = \prod_{k=1}^m e^{-i\tau_k \hat{B}} e^{-i\tau_k \|\hat{A}_k\| \hat{A}_k^{(\mathcal{N})}}$ for the N -system unitary as described. The protocols are thus sequences of these finite time periods τ_k , over which the system-bath interaction strengths of the qubits j are multiplied by the respective $\mu_j^{(k)}$. Specifically, we consider (a) a synchronous protocol, $\mu_j^{(k)} \equiv \mu^{(k)}$, generating an alternate product as mentioned above. An alternative is (b) an asynchronous protocol consisting of concatenations of 8-step cycles, where all subsets of the qubits are brought in contact with the bath as follows: over τ_1 , all qubits are detached from the bath, $\mu_j^{(1)} = 0$, after which qubit 1, then qubit 2, and then qubit 3 alone is brought in contact with the bath over τ_2, τ_3, τ_4 , respectively. After this, only qubits 1 and 2, then only qubits 1 and 3, and then only qubits 2 and 3 are brought simultaneously in contact with the bath over τ_5, τ_6, τ_7 , respectively. Finally, all 3 qubits are brought in contact with the bath over τ_8 . After this, at time $t_8 = \sum_{k=1}^8 \tau_k$, the cycle is repeated from the beginning over different times $\tau_9 \dots \tau_{16}$, and with different interaction strengths for those qubits in bath contact, and so forth, until a maximum number n of exponential factors in \hat{U}_{system} has been reached. The qubit-bath interaction strengths of those qubits j which are in bath contact over τ_k are adjusted synchronously, $\mu_j^{(k)} \equiv \mu^{(k)} > 0$ as in (a). In both protocols (a) and (b), we minimize the (squared) operator distance $D = \|\hat{U}_{\text{system}} - \hat{U}_F\|^2$ [36] of the implemented unitary \hat{U}_{system} and a given target \hat{U}_F by gradient descent. The minimum distance reached for a given number of exponential factors n is shown in Fig. 2 for the case where \hat{U}_F is either the CNOT (see Sec. IIIB below) or the Toffoli (i.e., C^2NOT) gate [37] for both types of protocols. The choice of a Toffoli gate is motivated by its relevance for quantum computing, but analogous results can be obtained for a random gate. Fig. 2 demonstrates that a desired target \hat{U}_F can generally be reached for both types of protocols as soon as n surpasses a threshold $[> (2^3)^2 = 64]$, albeit smaller n may suffice for specifically given \hat{H}_j , \hat{S}_j , and \hat{U}_F as in the closed expression for \hat{U}_{CNOT} mentioned above]. An example for each type of protocol is illustrated in Fig. 3. The choice of a synchronous protocol for the Toffoli gate and an asynchronous protocol for the CNOT gate is here irrelevant, since each gate can be generated with both types of protocols.

The asynchronous protocol requires a system-resolved switching of the coupling but exhibits a more robust convergence. This is illustrated in Fig. 4, where in contrast to Fig. 2, the three qubits and their couplings are identical, i.e., there is one single (randomly given) \hat{H} , so that the local winds (acting in the three-qubit Hilbert space)

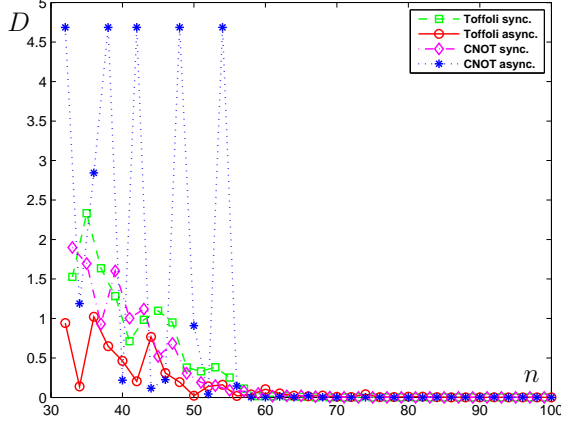


FIG. 2: Minimized squared operator distance $D = \|\hat{U}_{\text{system}} - \hat{U}_F\|^2$ as a function of the number n of exponential factors in \hat{U}_{system} for concatenated Zermelo navigations with three qubits $j = 1, 2, 3$ sharing a bosonic bath. The plots show the implementation of a $\hat{U}_F = \text{CNOT}$ gate between qubits 1 and 3 as well as a $\hat{U}_F = \text{Toffoli}$ gate, using the synchronous and the asynchronous protocol. The CNOT gate refers to the quantum repeater relay station depicted in Fig. 5. The local winds \hat{H}_j and couplings \hat{S}_j are randomly given for each individual j .

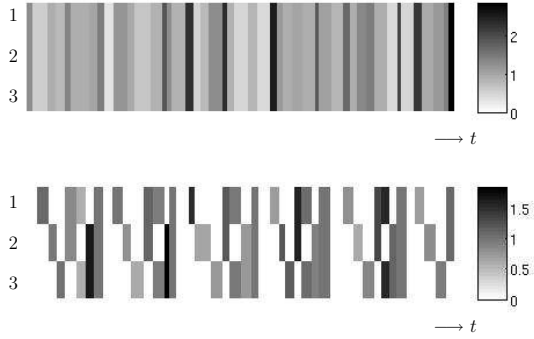


FIG. 3: Illustration of example protocols consisting of concatenated Zermelo navigations of qubits 1,2,3. Shown are the strengths $\|\hat{A}_k\|$ of the system-bath interaction (the navigator’s “engine power”) over the time intervals τ_k for those qubits in bath contact during τ_k . Upper plot: synchronous protocol generating a Toffoli gate ($n = 92$), lower plot: asynchronous protocol generating a CNOT gate between qubit 1 and 3 ($n = 84$). The CNOT gate refers to the quantum repeater relay station shown in Fig. 5.

are $\hat{H}_1 = \hat{H} \otimes \hat{I} \otimes \hat{I}$, $\hat{H}_2 = \hat{I} \otimes \hat{H} \otimes \hat{I}$, $\hat{H}_3 = \hat{I} \otimes \hat{I} \otimes \hat{H}$, and the local couplings \hat{S}_j are analogously given by one single (randomly given) \hat{S} . Fig. 4 demonstrates that the synchronous protocol fails in this case, in contrast to the asynchronous protocol.

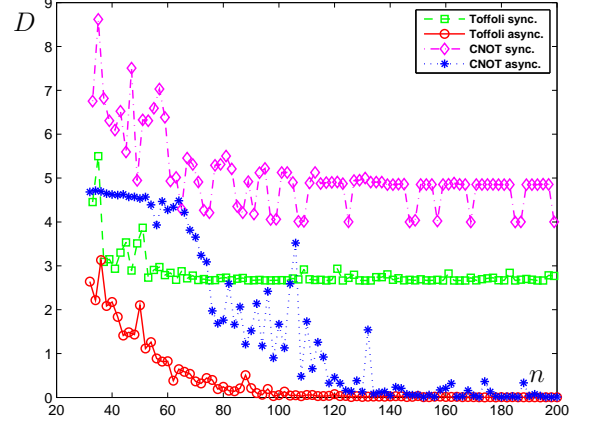


FIG. 4: Same analysis as in Fig. 2, except that the randomly chosen local winds \hat{H}_j and couplings \hat{S}_j are identical for each j .

B. Quantum repeater relay station

Depending on the application, the system coupling may follow a restricted topology. For example, the systems may be arranged in a chain, and bath-induced coupling is possible only for nearest neighbors. In the simplest case, three qubits form a linear array as shown in Fig. 5, with a presence of direct couplings κ_{12} and κ_{23} , but absence of an end-to-end coupling, $\kappa_{13} = 0$. This

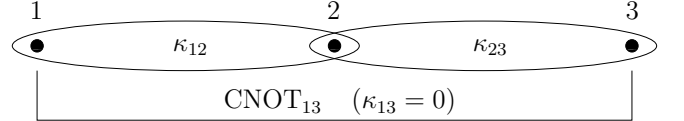


FIG. 5: Quantum repeater relay station: qubits 1 and 3 are coupled to qubit 2 but not to each other directly. The intermediate qubit 2 can be eliminated by concatenated Zermelo navigations to generate a direct coupling between qubits 1 and 3. An example protocol for generating a CNOT gate between these qubits is shown in the lower plot of Fig. 3. Analogous results can be obtained for a random gate.

setup resembles our original problem with the role of the bath adopted by the intermediate qubit 2. Similar to the elimination of the bath by Zermelo navigation, the intermediate qubit 2 can be eliminated by concatenated Zermelo navigations to generate a (qubit 2 - mediated) direct coupling between qubits 1 and 3 alone and thus a desired target gate $\hat{U}_F^{(13)}$. Since nesting of this procedure on larger time scales for a successive elimination of blocks of subsystems is (aside from the open system dynamics) reminiscent of quantum repeaters [38–40], we refer to the setup depicted in Fig. 5 as a quantum repeater relay station.

IV. GENERALIZATIONS

A. Motivation

Although concatenating Zermelo navigations as discussed in Sec. III is a straightforward option for reaching a target, similar to the construction of a quantum circuit from individual gates, the concatenations as a whole are no longer time-optimal, since they introduce refactorizations of the overall system-bath unitary at the intermediate times $t_m = \sum_{k=1}^m \tau_k$ as an artifact of this approach. The optimal system-bath trajectory would be a brachistochrone $\hat{U}(t)$ that starts at the identity and refactorizes only at the time of arrival. Even if we choose (concatenated) Zermelo navigations, we must ensure (a) the experimental realization of the required modulations $\hat{S}_j(t)$ (and their resets), (b) the presence of a linear coupling to a bosonic bath with the desired exclusive coupling to harmonics of a fundamental frequency ω (and its adjustment), and (c) detailed knowledge of all operators \hat{H}_j , \hat{S}_j , and coupling coefficients $\kappa_{jj'}$, along with an absence of operator noise and decoherence. The realization of these requirements can be challenging.

To avoid these problems, we now consider an alternative that relies on measurements alone, i.e., a “closed loop” - scheme. Rather than minimizing the squared operator distance D , we maximize the fidelity [41], which is in our context defined as an average overlap $F \equiv \overline{\text{Tr}(\hat{\rho}_{\text{out}} \hat{\Pi})}$. Here, a factorized system-bath input state $\hat{\rho}_{\text{in}} = |\Psi\rangle\langle\Psi| \otimes \hat{\rho}_{\text{B}}$ is transformed to $\hat{\rho}_{\text{out}} = \hat{U} \hat{\rho}_{\text{in}} \hat{U}^\dagger$ by a total system-bath unitary operation \hat{U} describing the device, after which an overlap is measured with a projector $\hat{\Pi} = \hat{U}_{\text{F}} |\Psi\rangle\langle\Psi| \hat{U}_{\text{F}}^\dagger$. This projector is determined by the respective input system state $|\Psi\rangle$ transformed by the desired target system unitary \hat{U}_{F} and defines the measurement device. $\overline{(\dots)}$ denotes the uniform average over all system states $|\Psi\rangle$. Since the fidelity can thus be estimated from repeated binary measurements with sampled input states $|\Psi\rangle$ and gradually improved, this approach is independent of any assumptions about or knowledge of the environment and its coupling to the system.

As an illustration, we consider again the task of implementing a Toffoli gate as shown in Fig. 6. In contrast to Fig. 2, the three qubits $j = 1, 2, 3$ (again with randomly given \hat{H}_j and \hat{S}_j) are now coupled to a common random environment (with randomly given \hat{H}_{B} and \hat{B}_j) rather than the described bosonic bath. Instead of Zermelo navigation, subsets of the qubits are brought in contact with the environment for fixed time periods $\tau_k \equiv \tau$ in a fixed order given by the asynchronous protocol described above. The controls are the system-environment interaction strengths (cf. the $\mu_j^{(k)}$ above) over these fixed time periods. (Equivalently, we may fix the system-environment interaction strength and tune the time periods τ_k instead.) Plot 1 in Fig. 6 shows a simulation of the gradual increase of the fidelity as it would be seen in

a control loop. For simplicity, we have assumed that dur-

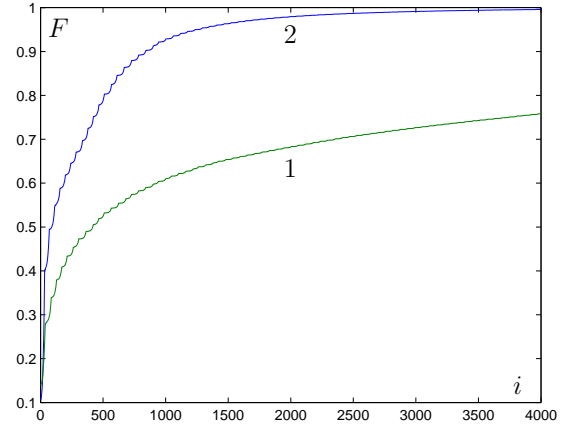


FIG. 6: Simulation of a closed loop gradient ascent of the fidelity $F = \overline{\text{Tr}[\hat{U}(|\Psi\rangle\langle\Psi| \otimes \hat{\rho}_{\text{B}}) \hat{U}^\dagger \hat{\Pi}]^{(\mathcal{P})}}$, which can be estimated experimentally from sampled binary measurements of $\hat{\Pi} = \hat{U}_{\text{F}} |\Psi\rangle\langle\Psi| \hat{U}_{\text{F}}^\dagger$, with the number of iterations i for three qubits sharing a random bath. Plot (1): \hat{U}_{F} = Toffoli gate acting on the whole three-qubit space, $\mathcal{P} = \mathcal{I}$, Plot (2): \hat{U}_{F} = Hadamard gate acting on a dynamically encoded decoherence free subspace $\mathcal{P} = \text{span}\{|000\rangle, |111\rangle\}$, using an asynchronous protocol with $n = 576$ time steps as controls.

ing each iteration i , a reliable estimate of the fidelity has been obtained by a sufficient number of measurements. Depending on the respective environment and the number of controls (time steps), such a control loop cannot guarantee to implement a desired gate in practice, as is illustrated in plot 1 of Fig. 6 for the Toffoli gate. Here, we have restricted ourselves to a 3-level ancillary system initially in a maximally mixed state $\hat{\rho}_{\text{B}} = \frac{1}{3} \hat{I}$ serving as environment, whose properties are unknown to the control loop. Since with an idealized Markovian bath (i.e., a bath in a narrower sense) no change of the fidelity would be observable by definition, such a control loop can at the same time serve to measure - in terms of the change of F - the deviation of the environment from an idealized bath and thus to witness the presence of non-Markovian properties. The latter application does not require the realization of a unitary with perfect (unit) fidelity.

B. Encoding a minimal noise subspace

To mitigate or remedy this controllability problem, we may restrict ourselves to input states $|\Psi\rangle \in \mathcal{P}$ belonging to a given subspace \mathcal{P} . The fidelity $F = \overline{\text{Tr}(\hat{\rho}_{\text{out}} \hat{\Pi})}^{(\mathcal{P})}$ is now given by the uniform average $\overline{(\dots)}^{(\mathcal{P})}$ with all states $|\Psi\rangle \in \mathcal{P}$ only. \mathcal{P} thus adopts the role of a minimal noise or decoherence free subspace [42, 43], which is dynamically encoded by the control loop, without knowledge of

any symmetries in the open system dynamics. Plot 2 in Fig. 6 illustrates this in the example of a Hadamard gate encoded in $\mathcal{P}=\text{span}\{|000\rangle, |111\rangle\}$. Due to the absence of any symmetries, the improvement of achievable fidelity is here due to the redundancy introduced by reducing the dimension of the (logical) quantum channel from 8 to 2.

The asynchronous protocol is here used merely in connection to Sec. III, where the piecewise constant μ_j are a consequence of the concatenated Zermelo navigations. It is obvious that instead of this, three independent continuous time-dependencies $\mu_j(t)$ can be applied just as well for the maximization of the fidelity, cf. e.g. [44].

V. SUMMARY AND OUTLOOK

In summary, we have applied the solution for the quantum Zermelo navigation problem to a scenario where a set of open systems share a bath. We have shown that Zermelo navigation allows to generalize an analytic case of system-bath factorization of the total unitary evolution operator to the presence of arbitrary background fields \hat{H}_0 , where only the individual system parts \hat{S}_j in the coupling operators have to be navigated in their respective local winds \hat{H}_j , whereas all bath operators remain unaltered. We have given a quantum state independent refactorization time limit in terms of the minimal bath energy. Finally we addressed the reachability of general target unitaries by concatenations of Zermelo navigations, making use of the two-exponential form of the individual Zermelo unitaries, and demonstrated the feasibility in numerical examples. In addition, to address the navigation to a general target in an unknown environment, we have considered a measurement-based closed loop scheme.

Interesting open questions we did not address and left for future work are the possibility of a symbolic solution for quantum Zermelo navigation in a time-dependent wind $\hat{H}_0(t)$ that generalizes (2). From the numerical point of view, the gradient-based analyses in Secs. III and IV are thought as illustrations of proof of concept only (not addressing the question whether a local search of a target unitary starting at the identity yields a time-

optimal navigation to this target unitary). For a comprehensive investigation it would be interesting to incorporate refined approaches such as subspace-selective self-adaptive differential evolution [30], Pareto front tracking [45], two-stage hybrid optimization [46], or an algebraic construction of the target unitary [47]. From the experimental point of view, devising a realization of the modulation (2) along with controlled interactions with an environment that obeys the desired bath properties is required. Potential candidates include Bose-Einstein condensates [27], trapped ions [48], atoms in an optical lattice [49], or arrays of optical cavities as depicted in Fig. 7.

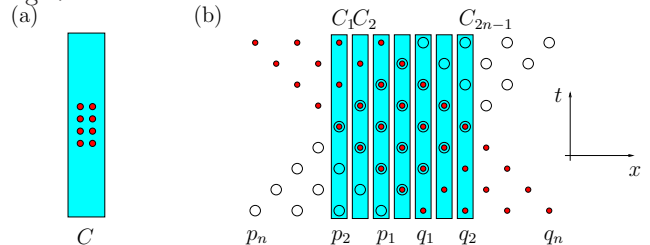


FIG. 7: Possible schemes for controlled bath coupling: (a) static setup in which the systems (circles) are arranged in an array confined by a cavity C ; (b) dynamic setup with two counterpropagating sequences of $n = N/2$ systems p (circles) and q (dots) which interact during coincident passages through an array of $2n - 1$ cavities: for identical interactions κ_{jk} within the cavities, the term $\sum_{j,k=1}^N \hat{S}_j \kappa_{jk} \hat{S}_k$ then reduces to a collective coupling $\sum_{j,k=1}^n \hat{p}_j \hat{q}_k = \hat{P} \hat{Q}$, where $\hat{P} = \sum_{j=1}^n \hat{p}_j$ and $\hat{Q} = \sum_{k=1}^n \hat{q}_k$ are variables of the respective system sequences. One may think of two trains of light pulses counterpropagating through a fiber, that may be regionally doped, or [(c), not shown] a ring resonator allowing repeated interactions.

Acknowledgments

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